Solution to 15.pdf

(1a.) False. Let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_4$. Let H be the subgroup $\mathbb{Z}_2 \oplus 0$ and K be the subgroup generated by $<(0,2)>$. Then $H \simeq K$ but $G/K \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $G/H \simeq \mathbb{Z}_4$. The quotient groups are not isomorphic.

(1b.) True. Let τ be a nontrivial element of S_n. Without loss of generality assume that $\tau(1) \neq 1$. Then, one has $s \neq 1$ such that $\tau(s) = 1$. Since $n \geq 3$, there exists $t \notin \{1, s\}$. Let $\gamma = (s,t)$. Then, $\tau * \gamma(s) = \tau(t) = 1 = \gamma(1) = \gamma * \tau(s)$. Note that $\tau(t) = 1$ will imply $t = 1$ which contradicts our choice of t.

(1c.) True. Let $G/Z(G) =< yZ(G) >$. For given $g, h \in G$ one has $gZ(G) = y^m Z(G)$ and $hZ(G) = y^n Z(G)$. This implies $g = y^m z_g$ and $h = y^n z_h$. Then $gh = y^m z_g y^n z_h = y^{m+n} z_g z_h = y^{n+m} z_h z_g = y^n z_h y^m z_g = hg$. Thus, G is abelian.

(1d.) True. By Cauchy's theorem since 3,7 are primes there exists $g, h \in G$ such that $o(g) = 3$ and $o(h) = 7$. *Claim:*$o(gh) = 21$. If $o(gh) = 1$ then, $g = h^{-1}$, *i.e.*, $h \in < g > \cap < h > = \{1\}$ by Lagrange's theorem. Thus, $g = h = 1$, contradiction. If $o(gh) = 3$ then as G is abelian one has $g^3 h^3 = 1 \implies g^3 = h^{-3} = 1$. But, $h^7 = 1$. So, $h = h^7 \cdot (h^{-3})^2 = 1$, contradiction. Similarly, if $o(gh) = 7$ one gets $g = 1$. Hence, $o(gh) = 21$ and G is cyclic.

(1e.) True. Every permutation of S_n can be written as product of disjoint cycles. Let $1 \neq \sigma \in S_n$ and $\sigma = (a_1, ..., a_r)(b_1, ..., b_t)...$ be decomposition of σ into product of disjoint cycles. Any cycle can be expressed as product of transpositions. So let, $(a_1, ..., a_r) = (a_1, a_r)(a_1, a_{r-1})...(a_1, a_2)$ be the first cycle expressed as product of transpositions. Now, the transposition $(a_1, a_j) = (1, a_1)(1, a_j)(1, a_1)$ for $j = r, ..., 2$. Thus, the assertion.

(2) Given $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. One has $A^2 = 1, B^2 = A^2$ and $B^{-1}AB = A^{-1}$. Now, the group generated by A and B with the above relations is the group of quaternions. Since $det(A) = det(B) = 1$, the above group is a representaion of quaternions in $SL(2, \mathbb{C})$. Order of this group is 8 and is notated $Q_8 = \{I, A, B, AB, A^*, B^*, (AB)^*, -I\}$ where $*$ is the conjugate-transpose operation. The subgroups of Q_8 are $< A >, < B >, < AB >$, the centre $\{1, -1\}$ and the nonproper subgroups $Q_8, \{1\}$. All these subgroups are normal in Q_8. Thq quotient of Q_8 with its centre has the cosets $\{\{1, -1\}, \{A, -A\}, \{B, -B\}, \{AB, -AB\}\}$. This group is isomorphic to the Klein 4-group as the relations $[A]^2 = [B]^2 = [AB]^2 = [1]$ and $[A][B] = [AB], [A][AB] = [B], [B][AB] = [A]$ holds true. The

subgroups $< A >, < B >$ and $< AB >$ are isomorphic to Z_4. The quotient group Q_8/Z_4 has order 2. Let us consider $Q_8/ < A >= \{[A], [B]\}$. One then has the relation $[A]^2 = [B]^2 = [A]$ and $[A][B] = [B][A] = [B]$. This implies that the quotient group is cyclic of order 2 and so $\cong Z_2$. Similarly, the subgroups $< B >$ and $< AB >$ also has quotient Z_2

(3a.) As $m \mid n$, let $n/m = r \in \mathbb{Z}$. Suppose $G =< a >$. Consider $H :=< a^r >$. We contend that H is a subgroup of order m. Indeed, if $x \in H$ then, $x = a^{rt}$ for some t and $x^m = a^{mrt} = a^{nt} = 1$. Conversely let $y^m = 1$ for $y \in G$, then $y = a^s$ for some s. This gives, $a^{ms} = 1$. Since $o(a) = n$ one has $n \mid ms$. Thus, $y = a^s = a^{r(sm/n)} \in< a^r >= H$. Hence $H = \{y \in G \mid y^m = 1\}$. Suppose K is another cyclic subgroup of G of order m. Let $K =< a^k >$ where k is the smallest integer such that $a^k \in K$. One has $a^{km} = 1 \Longrightarrow n \mid km \Longrightarrow km \geqslant n$. Further, m is the smallest integer such that $a^{km} = 1$ because $m = o(K) = o(a^k)$. Since $a^n = 1$, we get the other inequality $km \leqslant n$. Thus $km = n$ which implies $K = H$.

(3b.) Let $G = (\mathbb{Z}/2^n\mathbb{Z})^\times$. For $n = 3$, $G = \mathbb{Z}_8^\times = \{1, 3, 5, 7\}$. Since every element has order 2 one concludes that G is not cyclic for $n = 3$. Now for $n > 3$, one has $2^3 \mid 2^n$. We get a surjective homomorphism from $G \cup \{0\} \longrightarrow \mathbb{Z}_8$. This map induces group homomorphism from G onto $\mathbb{Z}_8^\times$. Since quotient of a cyclic group has to be cyclic and $\mathbb{Z}_8^\times$ is not cyclic, we conclude that G is not cyclic.

(4) Let $D_{2n} =< \{r, s \mid s^2 = e, r^n = e\} >$. One has $r^i s = sr^{-i}$ for $0 \leqslant i < n$. Let $\xi \in Z(D_{2n})$ then, $\xi r = r\xi$ and $\xi s = s\xi$. If $\xi = r^i s^j$ then one has $r^i s^j r = r^{i+1} s^j$. Left multiplication by r^{-i} gives $s^j r = rs^j$. As $o(s) = 2$ either $j = 0$ or $j = 1$. If $j = 0$ we get $r = r$, which is true. If $j = 1$ we get $sr = rs$, which is not true as D_2n is a nonabelian group for $n \geqslant 3$. Thus, $j = 0$. This implies $\xi = r^i$ and $r^i s = sr^i$. But, we have $r^i s = sr^{-i}$. This implies $sr^i = sr^{-i} \Longrightarrow r^{2i} = e$. Since, $o(r) = n$, n divides $2i$.

a. When n is odd and since $0 \leqslant i < n$ we conclude $i = 0$. Then, $\xi = r^i = e$ and $Z(D_{2n}) = \{e\}$.

b. When n is even $n = 2k$ and $2k$ divides $2i$ for $k = i$. Thus $\xi = r^i = r^k \in Z(D_{2n})$.

5(i) $GL(2, \mathbb{R})$ acts on $\mathbb{R}^2$ via multiplication then, the orbit of $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is given by $\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid A \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ for $A \in GL(2, \mathbb{R})\}$. Since $A[0, 0]^t = [0, 0]^t$ for every $A \in GL(2, \mathbb{R})$, one has orbit of $[0, 0]^t$ is $\{[0, 0]^t\}$.

5(ii) Suppose $[0, 0]^t \neq [a, b]^t \in \mathbb{R}^2$. Then either $a \neq 0$ or $b \neq 0$ or both. If both a and b are nonzero then

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Let, $A := \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$. If $a = 0$ then choose $A := \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}$ and is $b = 0$ choose

$A := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. Clearly, in all three cases $A \in GL(2,\mathbb{R})$ and hence any nonzero element of $\mathbb{R}^2$ lies in the orbit of $[1,0]^t$.

6(i) Let this action be denoted ϕ. Then $\mathrm{Ker}(\phi) = \{g \in G \mid \phi(g) = 1\} = \{g \in G \mid aH = gaH \ \forall\ a \in G\} = \{g \in G \mid a^{-1}ga \in H \ \forall\ a \in G\} = \{g \in G \mid g \in aHa^{-1} \ \forall\ a \in G\} = \bigcap\limits_{a \in G} aHa^{-1}$. Let $K = \bigcap\limits_{a \in G} aHa^{-1}$. Clearly, $K \lhd H$. Suppose N is a normal subgroup of G contained in H and containing K, *i.e.,* $K \subseteq N \lhd H$. If $x \in N \subset H$ then, $gxg^{-1} \in N$ for every $g \in G$. This implies $N \subset gHg^{-1}$ for every $g \in G$. Thus $N \subseteq K$. Hence, K is the largest normal subgroup of G contained in H.

6(ii) Consider the action of G on set X of all left cosets of H in G by left multiplication, *i.e.,* $g \mapsto (aH \mapsto gaH)$. This gives an homomorphism ϕ of G to S_n. If we let $K = \mathrm{Ker}\phi$ then K is normal subgroup of G with $[G:K] \cong \phi(G) \leqslant S_n$. Thus $[G:K]$ divides $n!$. Also, for $u \in K$, $\phi(u) = 1$ implies $g^{-1}ug \in H$ for every $g \in G$. This implies $K \subset H$.